Superspace Generalizations of Schouten-Nijenhuis Bracket

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Abstract

The Schouten-Nijenhuis bracket is generalized for the superspace case and for the Poisson brackets of opposite Grassmann parities.

I. Recently a prescription for the construction of new Poisson brackets from the bracket with a definite Grassmann parity was proposed [1]. This prescription is based on the use of exterior differentials of diverse Grassmann parities. It was indicated in [1] that this prescription leads to the generalizations of the Schouten-Nijenhuis bracket [2, 3, 4, 5, 6, 7, 8, 9, 10] on the both superspace case and the case of the brackets with diverse Grassmann parities. In the present report we give the details of these generalizations[1].

II. Let us recall the prescription for the construction from a given Poisson bracket of a Grassmann parity $\epsilon \equiv 0, 1 \pmod 2$ of another one.

A Poisson bracket, having a Grassmann parity ϵ, written in arbitrary non-canonical phase variables z^a

$$\{A, B\}_\epsilon = A \overleftarrow{\partial}_{z^a} \omega^{ab}_\epsilon(z) \overrightarrow{\partial}_{z^b} B, \tag{1}$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are right and left derivatives respectively, has the following main properties:

$$g(\{A, B\}_\epsilon) \equiv g_A + g_B + \epsilon \pmod 2,$$

$$\{A, B\}_\epsilon = -(-1)^{(g_A+\epsilon)(g_B+\epsilon)}\{B, A\}_\epsilon,$$

$$\sum_{(ABC)} (-1)^{(g_A+\epsilon)(g_C+\epsilon)}\{A, \{B, C\}_\epsilon\}_\epsilon = 0,$$

which lead to the corresponding relations for the matrix ω^{ab}_ϵ

$$g\left(\omega^{ab}_\epsilon\right) \equiv g_a + g_b + \epsilon \pmod 2, \tag{2}$$

$$\omega^{ab}_\epsilon = -(-1)^{(g_a+\epsilon)(g_b+\epsilon)}\omega^{ba}_\epsilon, \tag{3}$$

$$\sum_{(abc)} (-1)^{(g_a+\epsilon)(g_c+\epsilon)}\omega^{ad}_\epsilon \partial_{z^d}\omega^{bc}_\epsilon = 0, \tag{4}$$

where $\partial_{z^a} \equiv \partial/\partial z^a$ and $g_a \equiv g(z^a)$, $g_A \equiv g(A)$ are the corresponding Grassmann parities of phase coordinates z^a and a quantity A and a sum with a symbol (abc) under it designates a summation over cyclic permutations of a, b and c.

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[1]Concerning the generalizations of the Schouten-Nijenhuis bracket see also [11, 12].

The Hamilton equations for the phase variables z^a, which correspond to a Hamiltonian H_ϵ ($g(H_\epsilon)=\epsilon$),

$$\frac{dz^a}{dt}=\{z^a,H_\epsilon\}_\epsilon=\omega^{ab}_\epsilon\partial_{z^b}H_\epsilon \tag{5}$$

can be represented in the form

$$\frac{dz^a}{dt}=\omega^{ab}_\epsilon\partial_{z^b}H_\epsilon\equiv\omega^{ab}_\epsilon\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)}\stackrel{\rm def}{=}(z^a,d_\zeta H_\epsilon)_{\epsilon+\zeta}, \tag{6}$$

where d_ζ ($\zeta=0,1$) is one of the exterior differentials d_0 or d_1, which have opposite Grassmann parities 0 and 1 respectively and following symmetry properties with respect to the ordinary multiplication

$$d_0z^ad_0z^b=(-1)^{g_ag_b}d_0z^bd_0z^a,$$

$$d_1z^ad_1z^b=(-1)^{(g_a+1)(g_b+1)}d_1z^bd_1z^a \tag{7}$$

and exterior products

$$d_0z^a\wedge d_0z^b=(-1)^{g_ag_b+1}d_0z^b\wedge d_0z^a,$$

$$d_1z^a\tilde\wedge d_1z^b=(-1)^{(g_a+1)(g_b+1)}d_1z^b\tilde\wedge d_1z^a. \tag{8}$$

We use different notations $\wedge$ and $\tilde\wedge$ for the exterior products of d_0z^a and d_1z^a respectively.

By taking the exterior differential d_ζ from the Hamilton equations (5), we obtain

$$\frac{d(d_\zeta z^a)}{dt}=(d_\zeta\omega^{ab}_\epsilon)\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)}+(-1)^{\zeta(g_a+\epsilon)}\omega^{ab}_\epsilon\partial_{z^b}(d_\zeta H_\epsilon)\stackrel{\rm def}{=}(d_\zeta z^a,d_\zeta H_\epsilon)_{\epsilon+\zeta}. \tag{9}$$

As a result of equations (6) and (9) we have by definition the following binary composition for functions F and H of the variables z^a and their differentials $d_\zeta z^a\equiv y^a_\zeta$

$$(F,H)_{\epsilon+\zeta}=F\Big[\overleftarrow{\partial}_{z^a}\,\omega^{ab}_\epsilon\,\overrightarrow{\partial}_{y^b_\zeta}\quad+(-1)^{\zeta(g_a+\epsilon)}\,\overleftarrow{\partial}_{y^a_\zeta}\,\omega^{ab}_\epsilon\,\overrightarrow{\partial}_{z^b}$$
$$+\,\overleftarrow{\partial}_{y^a_\zeta}\,y^c_\zeta\left(\partial_{z^c}\omega^{ab}_\epsilon\right)\overrightarrow{\partial}_{y^b_\zeta}\Big]H. \tag{10}$$

By using relations (2)-(4) for the matrix ω^{ab}_ϵ, we can establish that the composition (10) satisfies all the main properties for the Poisson bracket with the Grassmann parity equal to $\epsilon+\zeta$. Thus, the application of the exterior differentials of opposite Grassmann parities to the given Poisson bracket results in the brackets of the different Grassmann parities.

By transition to the co-differential variables $y^{\epsilon+\zeta}_a$, related with differentials y^a_ζ by means of the matrix ω^{ab}_ϵ

$$y^a_\zeta=y^{\epsilon+\zeta}_b\omega^{ba}_\epsilon, \tag{11}$$

the Poisson bracket (10) takes a canonical form[2]

$$(F,H)_{\epsilon+\zeta}=F\left[\overleftarrow{\partial}_{z^a}\overrightarrow{\partial}_{y^{\epsilon+\zeta}_a}-(-1)^{g_a(g_a+\epsilon+\zeta)}\,\overleftarrow{\partial}_{y^{\epsilon+\zeta}_a}\overrightarrow{\partial}_{z^a}\right]H, \tag{12}$$

[2]There is no summation over ϵ in relation (11).

that can be proved with the use of the Jacobi identity (4).

The bracket (10) is given on the functions of the variables z^a, y^a_ζ

$$F=\sum_p\frac{1}{p!}y^{a_p}_\zeta\cdots y^{a_1}_\zeta f_{a_1\ldots a_p}(z),\qquad g(f_{a_1\ldots a_p})=g_f+g_{a_1}+\cdots+g_{a_p},$$

whereas this bracket, rewritten in the form (12), is given on the functions of variables z^a and $y^{\epsilon+\zeta}_a$

$$F=\sum_p\frac{1}{p!}y^{\epsilon+\zeta}_{a_p}\cdots y^{\epsilon+\zeta}_{a_1}f^{a_1\ldots a_p}(z),\qquad g(f^{a_1\ldots a_p})=g_f+\epsilon p+g_{a_1}+\cdots+g_{a_p}.$$

We do not exclude a possibility of the own Grassmann parity $g_f\equiv g(f)$ for a quantity f.

III. If we take the bracket in the canonical form (12), then we obtain the generalizations of the Schouten-Nijenhuis bracket [2, 3] (see also [4, 5, 6, 7, 8, 9, 10]) onto the cases of superspace and the brackets of diverse Grassmann parities. Indeed, let us consider the bracket (12) between monomials F and H having respectively degrees p and q

$$F=\frac{1}{p!}y^{\epsilon+\zeta}_{a_p}\cdots y^{\epsilon+\zeta}_{a_1}f^{a_1\ldots a_p}(z),\qquad g(f^{a_1\ldots a_p})=g_f+p\epsilon+g_{a_1}+\cdots+g_{a_p},$$

$$H=\frac{1}{q!}y^{\epsilon+\zeta}_{a_q}\cdots y^{\epsilon+\zeta}_{a_1}h^{a_1\ldots a_q}(z),\qquad g(h^{a_1\ldots a_q})=g_h+q\epsilon+g_{a_1}+\cdots+g_{a_q}.$$

Then as a result we obtain

$$\begin{aligned}(F,H)_{\epsilon+\zeta}\;&=\;\frac{(-1)^{[g_{b_1}+\cdots+g_{b_{q-1}}+(q-1)(\epsilon+\zeta)](g_f+g_l+p\zeta)}}{p!(q-1)!}\\&\times\;y^{\epsilon+\zeta}_{b_{q-1}}\cdots y^{\epsilon+\zeta}_{b_1}y^{\epsilon+\zeta}_{a_p}\cdots y^{\epsilon+\zeta}_{a_1}\left(f^{a_1\ldots a_p}\overleftarrow{\partial}_{z^l}\right)h^{b_1\ldots b_{q-1}l}\\&-\;\frac{(-1)^{(g_l+\epsilon+\zeta)(g_f+p\epsilon+g_{a_2}+\cdots+g_{a_p})+[g_{b_1}+\cdots+g_{b_q}+q(\epsilon+\zeta)][g_f+\epsilon+(p-1)\zeta]}}{(p-1)!q!}\\&\times\;y^{\epsilon+\zeta}_{b_q}\cdots y^{\epsilon+\zeta}_{b_1}y^{\epsilon+\zeta}_{a_p}\cdots y^{\epsilon+\zeta}_{a_2}f^{la_2\ldots a_p}\partial_{z^l}h^{b_1\ldots b_q}.\end{aligned}\tag{13}$$

Let us consider the formula (13) for the particular values of ϵ and ζ.

1. We start from the case which leads to the usual Schouten-Nijenhuis bracket for the skew-symmetric contravariant tensors. In this case, when $\epsilon=0$, $\zeta=1$ and the matrix $\omega^{ab}_0(x)=-\omega^{ba}_0(x)$ corresponds to the usual Poisson bracket for the commuting coordinates $z^a=x^a$, we have

$$\begin{aligned}(F,H)_1\;&=\;\frac{(-1)^{(q-1)(g_f+p)}}{p!(q-1)!}\Theta_{b_{q-1}}\cdots\Theta_{b_1}\Theta_{a_p}\cdots\Theta_{a_1}\left(f^{a_1\ldots a_p}\overleftarrow{\partial}_{x^l}\right)h^{b_1\ldots b_{q-1}l}\\&-\;\frac{(-1)^{g_f(q+1)+q(p-1)}}{(p-1)!q!}\Theta_{b_q}\cdots\Theta_{b_1}\Theta_{a_p}\cdots\Theta_{a_2}f^{la_2\ldots a_p}\partial_{x^l}h^{b_1\ldots b_q},\end{aligned}\tag{14}$$

where $\Theta_a\equiv y^1_a$ are Grassmann co-differential variables related owing to (11) with the Grassmann differential variables $\Theta^a\equiv d_1x^a$

$$\Theta^a=\Theta_b\omega^{ba}_0.$$

When Grassmann parities of the quantities f and h are equal to zero $g_f = g_h = 0$, we obtain from (14)

$$(F,H)_1 \stackrel{\text{def}}{=} (-1)^{(p+1)q+1} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F,H]^{a_2 \ldots a_{p+q}},$$

where $[F,H]^{a_2 \ldots a_{p+q}}$ are components of the usual Schouten-Nijenhuis bracket (see, for example, [8]) for the contravariant antisymmetric tensors[3]. This bracket has the following properties

$$[F,H] = (-1)^{pq}[H,F], \qquad \sum_{(FHE)} (-1)^{ps}[[F,H],E] = 0,$$

where s is a degree of a monomial E.

2. In the case $\epsilon = \zeta = 0$ and $\omega_0^{ab}(x) = -\omega_0^{ba}(x)$ we obtain the bracket for symmetric contravariant tensors (see, for example, [7])

$$(F,H)_0 \stackrel{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F,H]^{a_2 \ldots a_{p+q}},$$

where commuting co-differentials y^0_a connected with commuting differentials $y^a_0 \equiv d_0 x^a$ in accordance with (11)

$$y^a_0 = y^0_b \omega_0^{ba}$$

and the bracket $[F,H]^{a_2 \ldots a_{p+q}}$ has the following properties

$$[F,H] = -(-1)^{g_f g_h}[H,F], \qquad \sum_{(EFH)} (-1)^{g_e g_h}[E,[F,H]] = 0.$$

3. By taking the Martin bracket [13] $\omega_0^{ab}(\theta) = \omega_0^{ba}(\theta)$ with Grassmann coordinates $z^a = \theta^a$ ($g_a = 1$) as an initial bracket (1), we have in the case $\zeta = 0$ for antisymmetric contravariant tensors on the Grassmann algebra

$$(F,H)_0 \stackrel{\text{def}}{=} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F,H]^{a_2 \ldots a_{p+q}},$$

where the Grassmann co-differentials Θ_a related with the Grassmann differentials Θ^a as

$$d_0 \theta^a \equiv \Theta^a = \Theta_b \omega_0^{ba}.$$

The bracket $[F,H]$ has the following properties

$$[F,H] = -(-1)^{g_f g_h}[H,F], \qquad \sum_{(EFH)} (-1)^{g_e g_h}[E,[F,H]] = 0.$$

4. By taking the Martin bracket again, in the case $\zeta = 1$

$$d_1 \theta^a \equiv x^a = x_b \omega_0^{ba}$$

we obtain for the symmetric tensors on Grassmann algebra

$$(F,H)_1 \stackrel{\text{def}}{=} x_{a_{p+q}} \cdots x_{a_2} [F,H]^{a_2 \ldots a_{p+q}}.$$

The bracket $[F,H]$ has the following properties

$$[F,H] = -(-1)^{(g_f+p+1)(g_h+q+1)}[H,F], \qquad \sum_{(EFH)} (-1)^{(g_e+s+1)(g_h+q+1)}[E,[F,H]] = 0.$$

[3] Here and below we use the same notation [F,H] for the different brackets. We hope that this will not lead to the confusion.

5. In general, if we take the even bracket in superspace with coordinates $z^a = (x, \theta)$, where x and θ are respectively commuting and anticommuting (Grassmann) variables, then in the case $\zeta = 1$ we have

$$(F, H)_1 \stackrel{\text{def}}{=} y^1_{a_{p+q}} \cdots y^1_{a_2} [F, H]^{a_2 \dots a_{p+q}},$$

where

$$d_1 z^a \equiv y^a_1 = y^1_b \omega^{ba}_0.$$

The bracket $[F, H]$ has the following properties

$$[F, H] = -(-1)^{(g_f+p+1)(g_h+q+1)}[H, F], \qquad \sum_{(EFH)} (-1)^{(g_e+s+1)(g_h+q+1)}[E, [F, H]] = 0.$$

6. In the case of the even bracket in superspace as initial one with $\zeta = 0$ we obtain

$$(F, H)_0 \stackrel{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F, H]^{a_2 \dots a_{p+q}},$$

where

$$d_0 z^a \equiv y^a_0 = y^0_b \omega^{ba}_0.$$

The bracket $[F, H]$ has the following properties

$$[F, H] = -(-1)^{g_f g_h}[H, F], \qquad \sum_{(EFH)} (-1)^{g_e g_h}[E, [F, H]] = 0.$$

7. Taking as an initial bracket the odd Poisson bracket in superspace with coordinates z^a, for the case $\zeta = 0$ we have

$$(F, H)_1 \stackrel{\text{def}}{=} y^1_{a_{p+q}} \cdots y^1_{a_2} [F, H]^{a_2 \dots a_{p+q}},$$

where

$$d_0 z^a \equiv y^a_0 = y^1_b \omega^{ba}_1.$$

The bracket $[F, H]$ has the following properties

$$[F, H] = -(-1)^{(g_f+1)(g_h+1)}[H, F], \qquad \sum_{(EFH)} (-1)^{(g_e+1)(g_h+1)}[E, [F, H]] = 0.$$

8. At last for the odd Poisson bracket in superspace, taking as an initial one, we obtain in the case $\zeta = 1$

$$(F, H)_0 \stackrel{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F, H]^{a_2 \dots a_{p+q}},$$

where

$$d_1 z^a \equiv y^a_1 = y^0_b \omega^{ba}_1.$$

The bracket $[F, H]$ has the following properties

$$[F, H] = -(-1)^{(g_f+p)(g_h+q)}[H, F], \qquad \sum_{(EFH)} (-1)^{(g_e+s)(g_h+q)}[E, [F, H]] = 0.$$

Thus, we see that the formula (13) contains as particular cases quite a number of the Schouten-Nijenhuis type brackets.

IV. We give the prescription for the construction from a given Poisson bracket of the definite Grassmann parity another bracket. For this construction we use the exterior differentials with different Grassmann parities. We proved that the resulting Poisson bracket essentially depends on the parity of the exterior differential in spite of these differentials give the same exterior calculus [1]. The prescription leads to the set of different generalizations for the Schouten-Nijenhuis bracket. Thus, we see that the Schouten-Nijenhuis bracket and its possible generalizations are particular cases of the usual Poisson brackets of different Grassmann parities (12). We hope that these generalizations will find their own application for the deformation quantization (see, for example, [8, 14]) as well as the usual Schouten-Nijenhuis bracket.

We are sincerely grateful to J. Stasheff for the interest to the work and stimulating remarks. One of the authors (V.A.S.) sincerely thanks L. Bonora for the fruitful discussions and warm hospitality at the SISSA/ISAS (Trieste) where this work has been completed.

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